Approximating Pi with Series-Parallel Resistors

The following puzzle appeared in the February 1974 issue of *The American Mathematical Monthly*:[1]

> One is given an unlimited number of perfect 1-ohm resistors with which to construct a resistance of π ohms to within an accuracy of 10^{-6} ohms. Only series-parallel circuits are allowed. What is the minimum number of resistors necessary?

When I first read this puzzle I did not write it down, but remembered it as, "How many perfect 1-ohm resistors are needed to construct a series-parallel resistor network of π ohms to six decimal places accuracy?" Although the original puzzle was obviously worded with care, I think my paraphrase is a fair approximation.

Right away I confess that my answer to this puzzle (ca. 1974) was *not* minimal, in fact not even close. It did, however, contain the seed of an optimal solution. I said 26 resistors, while the proven minimum number of resistors necessary is 15. The following paragraphs describe my approach to the puzzle, after which I include a clip from the solutions issue of the journal, showing a 15-resistor solution.

Sometimes when facing a challenging problem it can be helpful to consider a simpler case of the same problem. For example, we might instead ask how many perfect 1-ohm resistors are needed ... for *two* decimal places accuracy?

In school we learned that the fraction $\frac{22}{7}$ approximates π to two decimal places. A few of the boys in my class believed that π was exactly equal to $\frac{22}{7}$, and were willing to put up their fists in defense their belief. $\frac{22}{7}$ was called an *improper* fraction back then. Possibly it still is. To make the fraction *proper* we were instructed to write $3\frac{1}{7}$.

Keeping in mind the puzzle that prompted this reminiscence, we see that 3 ohms plus $\frac{1}{7}$ ohms makes approximately 3.14 ohms or π ohms to two decimal places. However, that observation does not quite solve the problem, because we have not specified how to convert the fraction into a circuit consisting of 1-ohm resistors connected in some combination of series and parallel.

To make the fraction $\frac{1}{7}$ using only 1's, we could substitute the sum of seven 1's for 7 and write $\frac{1}{1+1+1+1+1+1+1}$. Nobody in his or her right mind would do such a thing under ordinary circumstances. How does that even help?

Resistors in series add:

$$R_{total} = R_1 + R_2 + R_3 + ... + R_n \tag{1}$$

[1] Proposed by Albert A. Mullin, U.S. Army Research Office, Arlington, Virginia.

While in parallel the total resistance is:

$$\frac{1}{R_{total}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + ... + \frac{1}{R_n} \tag{2}$$

Expression (2) can also be written:

$$R_{total} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_2} + ... + \frac{1}{R_n}} \tag{3}$$

Now I am going to do something silly. 1 is the same as $\frac{1}{1}$, so the fraction $\frac{1}{7}$ can be written:

$$\frac{1}{7} = \frac{1}{\frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1}} \tag{4}$$

The similarity of expressions (3) and (4) immediately suggests the fact that $\frac{1}{7}$ ohms can be made by connecting 7 1-ohm resistors in parallel. Moreover, connecting this parallel array in series with three 1-ohm resistors makes $3\frac{1}{7}$ ohms total resistance. Thus, no more than 10 resistors are needed to make π ohms to 2 decimal places accuracy, as Figure 1 illustrates.

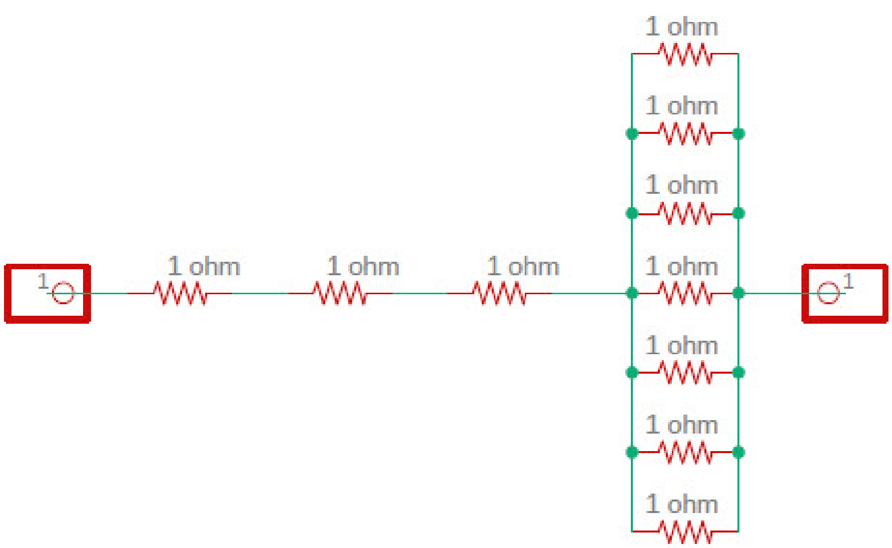


Figure 1: $\frac{22}{7}$ (approximately 3.14) ohms

At this point I am going to pull another fraction out of the air, although it didn't originally float down that way, as far as I know. Just as $\frac{22}{7}$ is the closest we can come to π with a 1-digit denominator, the fraction $\frac{355}{113}$ is the best we can do with 3 digits. After we fiddle a little with this fraction I will explain where it comes from.

The fraction $\frac{355}{113}$ evaluates to decimal 3.141592920353982301, a number that has the same first 6 decimal digits as the transcendental number π. If we could pull a similar trick with this fraction as we did with $\frac{22}{7}$, the original puzzle would be in hand.

Now a little magic...

$$\frac{355}{113} = 3 + \cfrac{1}{7 + \cfrac{1}{16}} \tag{5}$$

The expression on the right side of equation 5 is called a *continued fraction*. Let's compute the value to be sure that it agrees with the previously computed decimal value of the fraction. Starting at the bottom, $\frac{1}{16}$ is .0625. Add 7 and compute the reciprocal. $\frac{1}{7+.0625}$ is decimal .14159292035....

In general, it is possible to start with any decimal number, for example with a finite decimal approximation to π, and convert that decimal number to a continued fraction. The best rational approximations to the given number (called *rational convergents*) are obtained by evaluating the continued fraction to various depths.

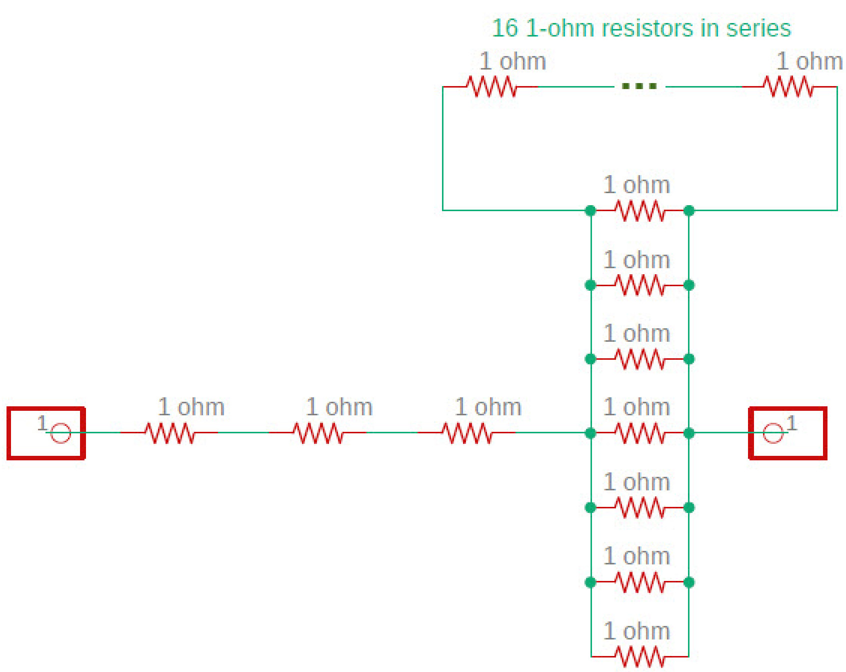


Figure 2: $\frac{355}{113}$ (approximately 3.141592) ohms

Back to resistors – Just as $3 = 1 + 1 + 1$ and $7 = 1 + 1 + 1 + 1 + 1 + 1 + 1$, so also the number 16 can be written as the sum of sixteen 1's. From these considerations we obtain the diagram above (Figure 2).

The puzzle asks for a number, not a diagram, so $3 + 7 + 16 = 26$ will be our soon to be bettered answer.

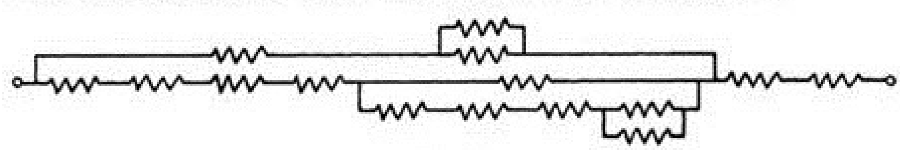
Postscript

As mentioned near the beginning of this write-up, the 'solution' outlined in the preceding paragraphs is not minimal. Various reader-submitted solutions were summarized in the February 1975 issue of *The American Mathematical Monthly* (one year after publication of the original problem).

One key solution among the several presented demonstrated that the rational convergent $\frac{355}{113}$ can be realized with 15 resistors, and no fewer than 15. My own insight that rational convergents lead toward the solution was correct, but I failed to imagine how rich a series-parallel resistor network could be.

The following scanned image (excerpted from the published solutions) presents a network of fifteen 1-ohm resistors that satisfies the problem statement. Evidently, 1975-era puzzle solvers did not have Autodesk Eagle or KiCad!

Studying the convergents to π using the continued fraction expansion of Wallis, we observe that the first convergent a/b which satisfies $|a/b - \pi| < 10^{-6}$ is $a/b = 355/113$. Now 355/113 can be realized by the 15-circuit of Figure 2.



No other fraction with numerator less than 2000 approximates π to within 10^{-6}, and since $f_{17} = 1597$, at least 16 resistors would be necessary to realize this or any other sufficiently close approximation. If we can show that 355/113 cannot be realized by any k-circuit with $k \leq 14$, we shall be done.

Figure 3: Excerpt *The American Mathematical Monthly*, February 1975.

The full proof that the number 15 is minimal has been cut from the excerpt, as it is more than one full journal page in length. Basically the proof addresses 3 sub-cases, proving each in turn to be impossible within the given constraints.