

# Microsoft Phi4

March 5, 2025

## 1 Problem

Problem: The ring-diode mixer circuit, such as found in superheterodyne radio receiver designs, is based on a trigonometric identity such as:

$$\cos \alpha \cos \beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$

which expresses a product in terms of a sum and difference. Please demonstrate this identity geometrically, or verify it from elementary (basic) identities.

To verify the trigonometric identity  $\cos \alpha \cos \beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$  using basic trigonometric identities, we can start by expressing cosine in terms of exponential functions. This approach utilizes Euler's formula:

$$e^{i\theta} = \cos \theta + i \sin \theta$$

From this, we have:

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Using this expression for cosine, we can write:

$$\begin{aligned}\cos \alpha &= \frac{e^{i\alpha} + e^{-i\alpha}}{2} \\ \cos \beta &= \frac{e^{i\beta} + e^{-i\beta}}{2}\end{aligned}$$

Now, consider the product  $\cos \alpha \cos \beta$ :

$$\cos \alpha \cos \beta = \left( \frac{e^{i\alpha} + e^{-i\alpha}}{2} \right) \left( \frac{e^{i\beta} + e^{-i\beta}}{2} \right)$$

Expanding this product, we get:

$$= \frac{1}{4} ((e^{i\alpha} + e^{-i\alpha})(e^{i\beta} + e^{-i\beta}))$$

$$= \frac{1}{4} \left( e^{i(\alpha+\beta)} + e^{i(\alpha-\beta)} + e^{-i(\alpha-\beta)} + e^{-i(\alpha+\beta)} \right)$$

Now, group the terms:

$$= \frac{1}{4} \left( e^{i(\alpha+\beta)} + e^{-i(\alpha+\beta)} + e^{i(\alpha-\beta)} + e^{-i(\alpha-\beta)} \right)$$

Recognize that each pair of terms is a cosine expression:

$$= \frac{1}{4} (2 \cos(\alpha + \beta) + 2 \cos(\alpha - \beta))$$

Simplify the expression:

$$= \frac{1}{2} (\cos(\alpha + \beta) + \cos(\alpha - \beta))$$

Thus, we have shown that:

$$\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$

This completes the verification of the identity using basic trigonometric identities and Euler's formula.

## 2 Clarification

I asked Phi4 to clarify the first step: Please give step-by-step for getting from Euler's formula to an expression for  $\cos(\theta)$  in terms of exponentials.

Certainly! We can derive an expression for  $\cos(\theta)$  using Euler's formula, which is given by:

$$e^{i\theta} = \cos(\theta) + i \sin(\theta).$$

Euler's formula also implies its conjugate:

$$e^{-i\theta} = \cos(-\theta) + i \sin(-\theta).$$

Using the even and odd properties of cosine and sine ( $\cos(-\theta) = \cos(\theta)$  and  $\sin(-\theta) = -\sin(\theta)$ ), we can rewrite this as:

$$e^{-i\theta} = \cos(\theta) - i \sin(\theta).$$

Now, to find an expression for  $\cos(\theta)$ , add the two equations together:

1. Start with:

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

2. And:

$$e^{-i\theta} = \cos(\theta) - i \sin(\theta)$$

3. Add these two equations:

$$e^{i\theta} + e^{-i\theta} = (\cos(\theta) + i \sin(\theta)) + (\cos(\theta) - i \sin(\theta))$$

4. Simplify the right-hand side:

$$e^{i\theta} + e^{-i\theta} = 2 \cos(\theta)$$

5. Solve for  $\cos(\theta)$ :

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Thus, the expression for  $\cos(\theta)$  in terms of exponentials is:

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

This derivation uses basic properties of complex numbers and trigonometric identities to express cosine as a combination of exponential functions.